#### Example: Equivalence classes

Let’s look at the amount M of students enrolled in the subjects medicine, computerM

science, and history. We have seen that the property of whether two students are in the same course of study forms an equivalence relation on . We assume that each course has at least one student (otherwise the university would not offer it). Each of the students is therefore enrolled in exactly one of these courses. Then we can define three sets:First, we note that SSSMGI: =: =: = xxSx∈∈∈ ≠∅MMMx studies computx studies history, x studies medicineS ≠ ∅ and S ≠ ∅er ⊆. This is important according to the previ-science⊆MM ⊆ M

ous definition, because equivalence classes are always non-empty subsets.M I G

property for equivalence classes is fulfilled for all three sets.same course. Thus If x, y ∈ is SM, then x ∼ yx and . The same applies to y are both studying medicine, i.e., they are enrolled in thex, y ∈ SI and x, y ∈ SG. Thus the first

Let us consider the equivalence relation course of study and in particular are both studying medicine. But that also means SIf sets x ∈ S, because S and M is equivalent to SSS is just the quantity of all medical students. The same applies to the, S and S y ∈ M are equivalence classes with respect to , i.e., if x ∼ y∼n applies, then x and [a] := {a + kn|k ∈y. □ are in the samey ∈ three sets.M I G. Thus the second property for equivalence classes is also fulfilled for allM

M I G on , which is explained as fol-

Because

and . It applies

. Because of it applies that

One can now ask the question whether an element could also be contained in more than−=i.e., m)n a + kn − y. We reformulate the equation and obtain [a] y ∈ [a]n|x = a + kn(a + kn) − y. x = a + kn. Furthermore, let ). Thus there is an y = a + kn − mn = a + (kn|n(k − k')y = a + k'n∼ □ nmn y,

Altogether it follows that is an equivalence class with respect to n. one equivalence class. The following theorem states that this is not possible.

If M is a set and R is an equivalence relation to M R M, then every x ∈ M is in exactly one Theorem: Uniqueness of equivalence classes

equivalence class of with respect to .

Let x ∈ M and be LxL := {y ∈ M |y ∼ x} is an equivalence class containing . x. Then we will prove that L Proof:

unique.First, we show that x x is Let unique, i.e., that Let Because y, y' ∈ L∼ is also transitive, and x. Then L ≠ ∅y' ∈ My ∼ x, because due to the reflexivity of . It applies that and x ∼ y'y ∼ y' and thus continues to follow . Because is symmetrical, it follows that xy' ∈ L∼. It remains to be shown that , x ∼y ∈ L is . x and therefore y ∼ y'. x ∈ Lx ∼ y'x ∼ yL .

It applies that xx.

Because L'y ∈ L∼x is transitive, it follows that Lx is an equivalence class containing is not in any other equivalence class.M R x x it applies that x .

Consequently, x x is

Let x be another equivalence class of with respect to , which contains .

x

class and because L'Let Let and because x. Thus y ∈ Ly' ∈ L'y' ∈ is Lx. Then it applies that xL ⊆ L'. Then it applies that x ∈ L'. Therefore, .x ∈ Lx, it follows with the second property of equivalence classes that L', it follows that with the second property of equivalence classes ⊆ Ly ∼ xy' ∼ x and therefore and also and also y ∈ ML = L'y' ∈ M. Because altogether. . Because L'x is an equivalence class□L is an equivalencey ∈

that x x x x x x x x

With this we have answered the question asked above: Each element of the set lies exactly in one equivalence class. Finally, let us briefly consider how equivalence classes can be related to each other.